

## Part III Predicate Logic

### Chapter 10 – Predicate Sentences and Predicate Proofs

#### 10.1 Introduction

As we have seen in the previous discussions, the key to assessing the formal validity of an argument is the proper translation of the sentences that are functioning as the premises and conclusion of the argument. However, it is not always easy to capture the subtleties of logical structure embedded in our ordinary language.

For example, a major liability of categorical logical notation is that it forces every sentence in ordinary language into being a sentence of class inclusion or exclusion. Ordinary language, however, is much richer than this. Sometimes we simply make statements like “The cat is on the mat.” Clearly, this assertion does not seem, without some forced stretches of the imagination, to be an assertion of class inclusion or exclusion.

Of special difficulty for categorical translation is what we have called singular sentences. Classic examples of such sentences, of course, are found in the most famous valid argument in philosophy:

*All human beings are mortal.*  
*Socrates is a human being.*  
*Therefore, Socrates is mortal.*

Clearly, the minor premise and the conclusion in this argument are singular sentences. As we have mentioned, categorical logic is limited when it comes to the proper symbolization of these two sentences. Recall that within the canons of categorical logic, the proper symbolization of “Socrates is a human being” cannot be captured with one of the standard form categorical sentences. This is because the claim here is about existence. To capture the assertion of existence in categorical logic, we are forced to express this singular sentence as a conjunction of the following two sentences: “All S are H” and “Some S are H.” Roughly translated, this would be something like, “All members of the ‘class’ of Socrates (which has at most just one member) are members of the class of human beings” and “There exists a member of the ‘class’ of Socrates that is also a member of the class of human beings.” This is awkward to say the least. In addition, to complicate things more, we must construct two syllogisms for each singular sentence in an argument in order to test the validity of the argument.

It was partially to address this problem that the more flexible system of sentential logical notation was developed. The fact is, however, that sentential logic does not fare much better than categorical logic when it comes to singular sentences, and indeed fares worse when it comes to expressing “quantity.” Let us see how this is so.

As we have learned, it is perfectly within the procedures of sentential logic to symbolize the singular sentence “Socrates is mortal” as a simple sentence “S.” Clearly, however, such a translation does not capture the full logical structure of the assertion. Indeed, the only way that sentential logic can express the full logical structure of such singular sentences is to express them as compound sentences. “Socrates is mortal” thus must be translated, given the possibility of empty classes, and given the assertion of existence in the sentence, as follows:

$$(S \supset M) \cdot S$$

We would read this as asserting, “If Socrates exists, then Socrates is mortal; Socrates does exist.” Again, this is an awkward construction of what seems to be a simple assertion.

If we do not make explicit the full logical structure of such singular sentences, then our argument, that certainly is valid, could not be assessed as valid by sentential procedures. Certainly, the following argument is not valid, but it might be considered a plausible translation of the argument if all of its premises and its conclusion were construed as simple sentences (“H”=“All human beings are mortal”; “S”=“Socrates is a human being”; and “M”=“Socrates is mortal”):

*H (Premise)*

*S (Premise)*

*M (Conclusion)*

Clearly “((H•S)⊃M)” is not tautological, since if “H” and “S” were true, and “M” were false, this conditional sentence that expresses this argument would be false. Hence, the argument it expresses is invalid.

If we try to express this argument within the canons of sentential logic, we run into all sorts of problems in trying to capture the embedded logical structure of its premises and conclusion. In fact, it is difficult to see how the first premise could be expressed adequately in sentential logic, since a major factor in its logical structure is the universal quantifier, “All.” In this regard, categorical logic has advantages over sentential logic because sentential logical notation has no symbolic resources for expressing quantification. The best we could hope for in expressing the major premise of our argument is something like: “If something is a human being, then it is mortal.” Symbolically expressed this would be “H⊃M.” Our argument might look something like the following:

*Premise:*  $H \supset M$  (If something is a human being, then it mortal.)

*Premise*  $(S \supset H) \cdot S$  (If Socrates exists, then Socrates is a human being; and Socrates does exist)

*Conclusion*  $(S \supset M) \cdot S$  (If Socrates does exist, then Socrates is mortal; and Socrates does exist.)

Now we can prove that this argument is valid as follows:

- 1)  $H \supset M$      P
- 2)  $(S \supset H) \cdot S$      P
- 3)  $S \supset H$      2 SIMP
- 4)  $S$      2 SIMP
- 5)  $S \supset M$      1, 3 HS
- 6)  $(S \supset M) \cdot S$      4,5 CONJ

Fortunately, the various problems with categorical and sentential notation are solved in predicate logic. These problems include: (1) expressing singular sentences; (2) capturing the richness of ordinary assertions, which often seem to go beyond mere class inclusion and exclusion; and (3) expressing quantity. In many ways this new system of logic combines the best of the symbolic notations in categorical and sentential logic. Let's now see how this new system of logical notation works.

## 10. 2 Predicate Symbolization

In logic, a sentence can be defined as having a basic subject-predicate structure. In logic, the only sentences that are candidates for making arguments are sentences that have a content that is either true or false. This gets refined as follows: the true/false content of a sentence is truly or falsely predicated of a subject. A model of this structure is something like: "Socrates is mortal." Here "Socrates" is the subject and "is mortal" is the predicate. As we might put it, in a sentence a predicate is what is (truly or falsely) "said of" a subject. Of course, not all sentences have this basic subject-predicate structure; but this is not surprising since not all sentences express sentences or contain a true/false content. For example, the directive sentence "Give me a beer" is not such a sentence in this sense because nothing is "said (truly or falsely) of some subject." For the same reason, "I promise to pay" does not express a true/false sentence either. Here the language is functioning as a performative and as such it can be successful or not, but not true or false.

As logicians, we are interested exclusively in true/false sentences, since they are the components of the arguments that we are assessing. We therefore need a logical notation that will adequately express this subject-predicate logical structure. That is, we need a way of symbolically representing the fact that every true/false sentence asserts that some subject

exemplifies some property. Again, think of the sentence “Socrates is mortal” as a model of such a predication. Here the property of mortality is attributed to Socrates (and if Socrates is not mortal, this property is falsely attributed to him. Indeed he professed to believe that he (his soul) was immortal.

The notation of predicate logic was developed to express this subject-predicate structure of sentences. In this notation, we use upper case letters to represent properties. We call these upper case letters **predicate letters**. (This is a marked departure from sentential logical notation where upper case letters are used to represent simple sentences.) Usually we select a letter—as we did in sentential logic—that reminds us of the particular content of the sentence we are symbolizing. For example, we might well use “M” to stand for the property “is mortal” in the sentence “Socrates is mortal.”

So all we need now is to find a symbolic notation for expressing the subject term in the subject-predicate assertion “Socrates is mortal.” It should be clear by now that this is a singular sentence. To symbolize the subject of this singular sentence, we will use lower case letters that remind us of what the subject is. Nevertheless, we must qualify this convention slightly. In representing singular subjects with lower case letters, we are going to use only the letters from “a” to “t.” We will reserve the letters “u” to “z” for other purposes that we will get to in just a moment. We call these representations of the subjects of our sentences **individual constants**. It is a good policy again to use letters, if available, that remind you of the subject that the letter is representing. Therefore, we would obviously use “s” for “Socrates.” These individual constants are placed after the predicate letter as a subscript.

Now we are ready to symbolize the singular sentence, “Socrates is mortal.” In predicate notation we do this by representing the subject with a lower case letter (an individual constant) and placed as a subscript to the predicate term, that is, the upper case letter. So we symbolize “Socrates is mortal” as follows: “M<sub>s</sub>.” (Now don’t think we are making some kind of statement about gender; it just looks like we have given Socrates a feminine title.) OK, let’s expand this formula to any singular sentence whatsoever. Consider the following examples:

*“Socrates is wise”* W<sub>s</sub>

*“Fido is barking”* B<sub>f</sub>

*“The Mayor is running for re-election”* R<sub>m</sub>

In predicate logic, as in sentential logic, we make a distinction between simple and compound sentences. “Socrates is mortal” is a simple sentence. As was true within the notation of sentential logic, simple sentences are used to form compound sentences in the notation of predicate logic. To form these compound sentences, we return to the convention of using the truth-functional symbols that were discussed in sentential logic (conjunctions, disjunctions, negations, conditions, and biconditionals). We can assert the following compound sentence,

“Either Socrates is mortal or he is immortal,” as, “ $M_s \vee \sim M_s$ .” And if we want to say, “If Socrates is a human being then he is mortal,” we can do so as follows: “ $H_s \supset M_s$ .” As in the case of sentential logical notation, these compound sentences in predicate notation are truth functional.

In sum, the notation of predicate logic is an extension of the notation of sentential logic. Both of these logical systems divide all sentences into either simple or compound; both systems use the same truth-functional connectives to form compound sentences; and both systems are truth-functional.

### 10.3 Quantifiers

In addition to being able to express the full logical structure of singular sentences, as well as the full logical structure of compound singular sentences, the notation of predicate logic can also express two other kinds of truth functional compound sentences. These two sentences are what categorical logic referred to as “universal sentences” and “existential sentences.” Here it is categorical logic that has the advantage over sentential logic, since the former has resources for expressing quantity (the quantifiers: “All,” “No,” and “Some”) and the latter does not. Again, predicate notation will combine the strength of categorical logic with the strength of sentential logic by providing the latter with a conventional way of representing quantity. Let's see how this would work by considering how predicate notation expresses universal and existential assertions.

Recall from categorical logic that the form of the universal sentence is “All S are P” or “No S are P.” Now, let the universal affirmative expression “All S are P” represent the universal sentence “All swans are white” and the universal negative expression “No S are P” represent the expression “No swans are white.” The first sentence asserts that every member of the subject class of swans exemplifies the same property of being white and the second asserts that no member of the subject class exemplifies the property of being white. (Of course, both of these assertions are false.)

To express these universal assertions in sentential logic, we use conditional sentential forms, since what is claimed in the first sentence is that being a swan is a sufficient condition for being white and what is claimed in the second assertion is that being a swan is a sufficient condition for not being white. We symbolize these conditional relations as: “ $S \supset W$ ” and “ $S \supset \sim W$ .”

What is missing from these sentential expressions is a representation of quantity. The advance of predicate notation is that it provides us with a convention for expressing quantification. The way that this notation expresses the universal quantifier (“All” or “No”) is by placing any one of the last three letters of the alphabet in the lower case, “x,” “y” and “z” within parentheses and letting this expression range over the subject-predicate expression it is quantifying. Universal quantifiers thus will look like one of the following (x), (y), or (z). We read such expressions as follows: (x) means “for any x,” (y) means “for any y” and (z)

means “for any z.” Using these conventions, we get the following predicate translations of our two universal sentences:

“All swans are white” is symbolized in predicate notation as  $(x)(Sx \supset Wx)$  and is read as, “For any x, if x is a swan then x is white.” Similarly, the expression “No swans are white” is symbolized as  $(x)(Sx \supset \sim Wx)$  and is read as, “For any x, if x is a swan then x is not white.” With these models, we should be able to symbolize any universal sentence.

In quantified predicate sentences, **individual variables** rather than **individual constants**. We use the last three letters of the alphabet to represent these variables. In a given sentence, these letters will correspond to the letter used in the quantifier. Consider the assertion, “All human beings are mortal.” In predicate notation, we symbolize this sentence as:  $(x)(Hx \supset Mx)$ . We read the “x” in “Hx” and “Mx” as standing for every member of the class of all “x’s.” This expression does not refer to any particular member of the class of human beings, say “Socrates” but to any member. So, the full expression is read as follows: “For any x, if x is a human being, then x is mortal.”

The second thing to notice is that the quantifier in this system of notation has the same kind of range as the negation sign ( $\sim$ ). There is a major difference between “ $\sim (Hs \bullet \sim Ms)$ ” and “ $Hs \bullet \sim Ms$ ”. The first may represent the following assertion: “It is false that Socrates is both a human being and not mortal.” We read the second expression as asserting, “Socrates is a human being and Socrates is not mortal.” These two expressions are clearly different since the first seems obviously true and the second obviously false. Similarly, there is a major difference between the assertion  $(x)(Hx \supset Mx)$  and the assertion  $(x)Hx \supset (x)Mx$ . The first assertion is read, “For any x, if x is a human being, then x is mortal.” The second assertion is read: “If everything is a human being, then everything in the universe is mortal. Clearly, these are quite different assertions.

The third thing to notice is that the expressions that quantifiers modify are not assertions independently of their quantification. Consider  $(x)(Hx \supset Mx)$ . If we take the quantifier away, what remains, namely, “ $Hx \supset Mx$ ,” is not a sentence. Rather, it is what we call a **sentential function**. These functions do not have any particular content, and neither do they assert anything; hence, they have no truth-value. The variables in such sentential functions (the ‘x’s’) in this case, are not governed by a quantifier and so are called **free variables**. When a quantifier is added, these variables become **bound variables**. When the variables in a sentential function get bound by a quantifier, then the function becomes a sentence; that is, it becomes an assertion that is either true or false. In the example above,  $(x)(Hx \supset Mx)$ , the x in “Hx” and the x in “Mx” are bound. In “ $Hx \supset Mx$ ” the x’s are both free. This distinction between free and bound variables will be important when we get to the rules of inference for predicate logic.

Now we must turn to existential sentences. Recall again from categorical logic that existential sentences, affirmative and negative, use “Some” as their common quantifier. “Some” here is translated as, “there exists at least one.” So the sentences “Some swans are

white” and “Some swans are not white,” both assert that there are some swans. Since swans do exist, then both of the sentences cannot be false. However, this would not be true in the case of the following two sentences: “Some unicorns are friendly” and “Some unicorns are not friendly.” Both of these sentences are false since they both assert falsely that unicorns exist.

To translate existential assertions into predicate notation, we need what we can call an existential quantifier. As we adopted the convention of using the last three letters placed within a parenthesis to represent a universal quantifier, we adopt the convention of using a backwards “E” ( $\exists$ ) plus one of these last three letters to represent an existential quantifier. So if we want to assert that, “Some swans are white” we notice that we not only want to assert that swans exist but also that they are white. This sentence, then, does not assert a conditional relation, but a conjunction. Again, we want to assert that swans exist and that they are white. So our symbolic representation for this in predicate notation is as follows: “ $(\exists x)(Sx \bullet Wx)$ .” We read this as, “There exists an x, such that x is a swan and x is white.”

If we want to assert an existentially quantified negative sentence, we can do so in a similar way. Suppose we want to express in predicate notation the sentence, “Some swans are not white.” We can do so by again using an existential quantifier that ranges over a conjunction. But in this case the right hand conjunct is negated. Our symbolic translation of this negative existential assertion then is as follows: “ $(\exists x)(Sx \bullet \sim Wx)$ .”

We can repeat these patterns and symbolize any standard form universal or existential, affirmative or negative, sentence. As we discovered earlier in our efforts to translate ordinary language into categorical and sentential notation, many assertions in ordinary language are not in “standard forms”. To make these translations, we adopted some techniques for translation that we can also use in translating ordinary assertions into predicate notation. I will mention some of these before we turn to a discussion of proofs in predicate logic.

As we have already noticed, sometimes assertions in ordinary language do not have explicit quantifiers, yet are implicitly quantified. Consider the following sentence, “Flowers bloom in spring.” When we try to figure out what is intended in this assertion, it should be clear that this assertion has an embedded existential quantifier in it. That is, what is asserted here is, “Some flowers bloom in spring”. So, we would symbolize this sentence in the notation of predicate logic as follows: “ $(\exists x)(Fx \bullet Bx)$ .” Here are some other examples. (Remember that universal sentences are symbolized with a universally quantified conditional sentence and existential sentences are symbolized with an existentially quantified conjunction.)

1. “Cheetahs run fast.”  $(x) Cx \supset Fx$
2. “There are Cheetahs in the zoo.”  $(\exists x)(Cx \bullet Zx)$
3. “A Cheetah is a mammal.”  $(x) Cx \supset Mx$
4. “A Cheetah has escaped.”  $(\exists x)(Cx \bullet Ex)$

It is important to use the correct quantifier in our translations. We can see this if we try to translate the false sentence “No cats are mammals” by using an existential quantifier as follows:  $(\exists x)(Cx \bullet \sim Mx)$ .

The replacement equivalences apply in predicate logic. For example, the equivalence known as material implication (IMP) tells us that the sentence  $(x)(Cx \supset Mx)$  is identical to  $(x)(\sim Cx \vee Mx)$ . As well, De Morgan’s Theorem (DM) allows us to substitute equivalent expressions of neither nor. We read  $(\exists x) \sim(Cx \vee Mx)$  as “There exists an x such that it is neither C nor M.” This is logically equivalent to  $(\exists x)(\sim Cx \bullet \sim Mx)$ .

There are as well some interesting equivalences between universal and existential sentences. Any existentially quantified sentence is logically equivalent to a corresponding negated universally quantified sentence and any universally quantified sentence is logically equivalent to a corresponding negated existential sentence. These logical equivalences are as follows, and will be useful when we turn to predicate proofs. (Here we are using a generic predicate “G.”)

<b>Quantifier Negation (QN)</b>
$\sim(\exists x)\sim Gx \equiv (x)Gx$
$\sim(x)\sim Gx \equiv (\exists x)Gx$
$\sim(\exists x)Gx \equiv (x)\sim Gx$
$\sim(x)Gx \equiv (\exists x)\sim Gx$

Putting these equivalences into English and using the predicate “is gifted,” we get the following translations:

1. *“It is false that some people are not gifted”  $\equiv$  “Everyone is gifted”*
2. *“It is false that everyone is not gifted”  $\equiv$  “Some people are gifted”*
3. *“It is false that some people are gifted”  $\equiv$  “Everyone is not gifted”*
4. *“It is false that everyone is gifted”  $\equiv$  “Some people are not gifted”*

When we turn to proofs in predicate logic, we will return to these logical equivalences, as they will be needed in our deductions in the same way that our equivalence rules in sentential logic were used in sentential proofs. For now, we will simply say that these rules can be referenced in a deduction with the abbreviation “QN” standing for “quantification negation.”

However useful predicate notion is, there are still some difficulties. Consider first what we have called exceptive sentences. These sentences involve phrases like, “All but,” or “All except.” As we pointed out previously, such sentences seem to be best translated as a conjunction of two sentences. For example, consider the following assertion: “All students, except those who are graduating seniors, are admitted to the party.”

This would seem to be best translated as a conjunction of two conditional sentences as follows:  $(x)(Gx \supset \sim Px) \cdot (x)(\sim Gx \supset Px)$ . In fact this symbolization is perfectly acceptable for exceptive sentences. However, a different and simpler translation is also correct. This involves the biconditional and the translation is as follows:  $(x)(Px \equiv \sim Gx)$ . We read this translation as: "Any student can go the party if and only if that student is not a graduating senior."

Many assertions in ordinary language are what we have called exclusive sentences. Such sentences sometimes involve the use of the word "only" or the phrase "only if." These sentences are obviously conditional and what "only" and "only if" modify are meant to be the necessary conditions in the "if/then" conditional relation. That is, what "only" and "only if" modify goes in the "then" or consequent place in the symbolic expression of the conditional sentence. Moreover, such sentences carry an implicit universal quantification. So for example, interpreting the sense of the following sentence, "Only seniors are admitted," we see that being a senior is a necessary condition and that the quantification is universal. Our translation into predicate notation would look like this:  $(x)(Ax \supset Sx)$ .

Again, do not let the placement of the "only" or "only if" throw you off. The following assertion is symbolized exactly as the one above: "You will be admitted only if you are a senior." Remember that if the word "if" occurs in the sentence you are attempting to translate, regardless of where it occurs in the syntax of the sentence, what it modifies is what we have called the sufficient condition in the "if/then" relation; hence, it goes in the "if" or antecedent place in the symbolic conditional expression. Again we read such conditionals as embedding a universal quantifier. So, for example, we read the assertions, "You will be admitted to the party if you are a senior" and "If you are a senior you will be admitted to the party" as synonymous and as meaning that being a senior is a sufficient condition for being admitted to the party. Our symbolic translation for both of these sentences then would be as follows:  $(x)(Sx \supset Ax)$ .

Along the same lines, if we are translating sentences with the words "necessary" or "sufficient" in them, we must be sure that we place necessary conditions as the consequents of conditional sentences and sufficient conditions as the antecedents of conditional sentences. For example, if someone says, "It is necessary to come to logic class in order to make a good grade," we symbolize that assertion as follows:  $(x)(Gx \supset Cx)$ .

Similarly, if someone asserts "Coming to logic class is all you need to make a good grade in the class," the sense of this assertion is that coming to class is sufficient for making a good grade in the class, and would be symbolized as:  $(x)(Cx \supset Gx)$ . And of course if one were to assert that "Coming to class was both necessary and sufficient for making a good grade in class," we would symbolize the claim as a biconditional as follows:  $(x)(Cx \equiv Gx)$ .

There is one more use of the word "only" to which we must alert. When, "the only" is used, such sentences carry an implicit universal quantification. Like other uses of "only,"

they may occur in the middle of the sentence that is being translated, as well as at the beginning. When “the only” occurs in the middle of a sentence, just rearrange the sentence so that “the only” comes first, and then translate it with the universal quantifier. The following two examples should make this clear, as both have the same translation:

- “*The only people who are allowed are women.*”  $(x) (Ax \supset Wx)$
- “*Women are the only people allowed.*”  $(x) (Ax \supset Wx)$

A further complication in sentences with “only” in them is that sometimes “only” modifies a conjunction and the proper translation of this is as a disjunction. For example, translate “Only juniors and seniors are admitted to the party” as:  $(x) (Ax \supset (Jx \vee Sx))$ .

It should be obvious that if we were to translate the consequent of the universally quantified conditional as a conjunction, this would mean that a necessary condition for being admitted is that a person be both a junior and a senior. This would be an absurd requirement, since one is either a junior or a senior and not both. There are other such uses of “and” that require a translation as “either/or.” For example, translate the sentence, “Hurricanes and tornadoes are dangerous storms,” as,  $(x)((Hx \vee Tx) \supset Dx)$ .

Sometimes the use of “and” is required when there is not one used explicitly, as well as when there is one used explicitly. Take the following example: “Venomous snakes are dangerous and potentially deadly.” The proper translation is:  $(x)((Sx \bullet Vx) \supset (Dx \bullet Px))$ .

Things get even more complicated when we mix “and” and “or” in the same sentence. Consider, for example, the following: “Juniors and seniors are exempt from finals if they have an A or B average.” Translate as:  $(x)[(Jx \vee Sx) \supset ((Ax \vee Bx) \supset Ex)]$ .

As well as mixing “and” and “or,” many sentences mix singular, universal and existential assertions together. We must pay close attention to these mixes if our symbolizations are to capture the meaning of the expressions we are trying to translate. The following examples should make it clear how some of these translations should be made.

- Either everything is connected or John Muir was mistaken:  $(x) Cx \vee Mm$
- If Socrates is wise, then one philosopher is wis:  $Ws \supset (\exists x)(Px \bullet Wx)$
- If some ancient Greek philosophers were wise, then some ancient Greeks were admirable:  $(\exists x)(Px \bullet Wx) \supset (\exists x)(Gx \bullet Ax)$
- If anything is possible, then something is possible:  $(x)Px \supset (\exists x)Px$
- Philosophers are wise unless they think they know everything:  $(x)[Px \supset (Wx \vee Kx)]$

You might be wondering at this point, “Can things get any more complicated? Well given the richness of ordinary language, the answer is yes they can. But this is about as far as we are going to take you in this introductory course. If you choose to move on to a more advanced course, there will be quite a few new techniques to master when it comes to translating the full logical structure of an assertion in ordinary language. Just to whet your appetite, we will mention a couple of these.

A little reflection will tell you that relational assertions pose a particular problem for the system of logical notation that has been introduced thus far. A relational assertion is something like, “John is smarter than Patrick.” In this assertion, we have a relational predicate (smarter than) and we can use a predicate letter to represent it. (Naturally, we would choose the letter “S”.) What makes this assertion different is that there are two individuals that bear this relation “S” to one another. Since we are dealing with two individuals, the logical structure of the assertion can be captured in predicate notation without having to worry about quantifiers. Here is how we do it: “Sjp”. We use the individual constants “j” to name John and “p” to name Patrick. This pattern should be easy to apply to other relational sentences.

Relational sentences do not always involve a relation between two individuals, and sometimes they don’t involve any individuals. Accordingly, we have to use quantifiers. The following examples may give you an idea of how we proceed here:

“Socrates knows nothing.”  
 $(x)(\sim Ksx)$   
 “If Socrates knows nothing, then he knows something □”  
 $(x) (\sim Ksx) \supset ((\exists y)(Ksy))$

In sentences like “Everything comes from something,” we are presented with a need for what is called overlapping quantifiers. All that this means is that we have more than one quantifier ranging over one sentenceal function. This can be two or more universal quantifiers, two or more existential quantifiers, or two or more of a mix of the two. We will end this section with just a few examples of how this technique works □

“Everything comes from something”  $(x) (\exists y)(Cxy)$   
 “Something comes from something”  $(\exists x) (\exists y) (Cxy)$   
 “Everything is related to everything else”  $(x) (y) (Rxy)$   
 “Something is related to everything”  $(\exists x) (y) (Rxy)$

I am going to have to leave it to future occasions to say anything further about the issue of translating ordinary language into predicate notation. It should be obvious to you that there is much more to consider. But for now, you have in hand ample techniques for

translating most of what we normally say into predicate notation. It is now time to move on to predicate proofs.

### 10.4 Predicate Proofs

Proving validity in predicate logic builds on the rules of inference and replacement equivalences that we used in sentential logic. But these rules are not sufficient for predicate deductions since the notation of predicate logic involves quantifiers. If we were dealing only with sentences that involve individual constants, the rules of sentential logic would be sufficient for predicate proofs. For example, consider the following argument:

*If Socrates is a human being, then he is not a god  
He is a human being  
Therefore, Socrates is not a god*

Our proof of this argument would look like this:

- 1)  $Hs \supset \sim Gs$     P
- 2)  $Hs$             P
- 3)  $\sim Gs$             1, 2 MP

As this proof shows, the rules of inference in sentential logic are sufficient for deductions that involve only predicates and individual constants  $\square$

Deductions that involve quantified sentences require additional rules. We have already introduced one of these rules, the rule of Quantifier Negation. This rule has four different expressions. To refresh your memory, here is the table of four ways this rule can be expressed.

Quantifier Negation (QN)
$\sim(\exists x)\sim Gx \equiv (x)Gx$
$\sim(x)\sim Gx \equiv (\exists x)Gx$
$\sim(\exists x)Gx \equiv (x)\sim Gx$
$\sim(x)Gx \equiv (\exists x)\sim Gx$

QN allows us to replace expressions of universal quantification with logically equivalent expressions of negated existential quantification. As well, it tells us that we can replace expressions of existential quantification with expressions of negated universal quantification. As a rule of replacement, QN allows us to go back and forth across the triple bar without affecting the truth-value of the assertion. The most important thing to notice about this rule is that it allows us to get rid of the negation sign when it precedes a quantifier. This move is essential when it comes to applying the four new rules of inference that we

must now add to the sentential rules of inference if we are to be able to make deductions from quantified sentences.

Before we introduce these four quantificational rules of inference, let us just say a word about why they are necessary for proofs in predicate logic. Since the first 9 rules of inference and the 10 standard equivalences for the use with the rule of replacement in sentential logic apply only to whole unquantified sentences, we must remove the quantifiers if these rules are to be applied. We have two rules for eliminating quantifiers. We will call these **instantiation rules**. Given that we might be attempting to derive a conclusion that is quantified, we must also have a way of restoring quantifiers. We have two rules for doing this and they are called **generalization rules**. It should become clear how these rules work in a deduction as we introduce each rule.

## 10.5 The Quantification Rules

1. The first quantificational rule of inference is called **Universal Instantiation (UI)**. Consider the claim, "Everything is physical." In predicate notation, this sentence is symbolized as follows: " $(x)Px$ ." Since the claim here is that everything in the universe exemplifies the property "P" (is physical), it follows that any particular instance of an individual in the universe must also exemplify this property. As such, if " $(x)Px$ " is true, then clearly " $Pr$ " or " $Ps$ " or " $Pm$ ," etc. (particular instances of individuals in the universe) must also be true. We call "r," "s," and "m" **individual constants, or names.**)

In this case these letters are individual constants, but they need not be. Clearly if any particular individual in the universe exemplifies the property "P" then any randomly selected individual will also exemplify "P." To express this implication, we can use the lower case letter "u" to represent any **randomly selected individual**. Here "u," "v," or "w" serve a very special function and we will call them **free variables**. We may say then that " $Pu$ " or " $Pv$ " follows from " $(x)Px$ ." As well, from " $(x)(Px \supset Sx)$ " UI allows us to deduce the sentential function,  $Pu \supset Su$ . With the universal quantifier eliminated, we can now apply the sentential rules of inference.

In summary, the most general statement of the rule of Universal Instantiation (UI) is as follows:

<p><b>Universal Instantiation (UI)</b></p> <p><math>(x)(Gx \therefore Ga)</math></p> <p><math>(x)Gx \therefore Gu</math></p> <p><i>"a" is an individual constant and "u" is a free variable</i></p>
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2. The second quantification rule is called **Universal Generalization (UG)** and can be stated as follows:

## Universal Generalization (UG)

$Gu \therefore (x)Gx$

*“u” is a free variable. Note this rule does not allow the inference:  $Ga \therefore (x)Gx$ , where “a” is an individual constant*

UG allows us to restore a universal quantifier to any sentential function “Gu.” It does not allow us to infer “Ga.” In other words, this rule tells us that it is a valid inference to move from the sentential function “Ru•Bu” to the universally quantified sentence “(x)(Rx•Bx).” However, UG does not allow us to infer this same sentence from “Rs•Bs.” The reasoning for this is not hard to follow. In the first case, since “u” is a free variable and thus stands for any randomly selected individual, it follows that if “u” exemplifies the properties “R” and “B,” then anything (any “x”) in the universe must also exemplify these properties. In the second case, it does not follow from the fact that some particular individual in the universe exemplifies these properties that everything does.

If any randomly selected American is patriotic, it follows that all Americans are patriotic. The symbolic representation of this inference is as follows: if “ $Au \supset Pu$ ” is assumed, where “u” is a free variable, then it follows that “ $(x)(Ax \supset Px)$ ” must be true. However, assuming that John is an American and is patriotic does not entitle us to infer (validly) that every American is patriotic. From “ $Aj \supset Pj$ ” it does not follow that “ $(x)(Ax \supset Px)$ .” The rule of UG does not allow this inference, even though the rule of UI does allow us to infer from the sentence “ $(x)(Ax \supset Px)$ ” the sentence “ $Aj \supset Pj$ .” What this shows is that UG is not simply a reversal of UI nor vice versa.

Now let’s put these two rules into practice with a deduction that will require that we use both rules. Our argument is as follows:

*If everyone is truly happy, then it is false that everyone is self-deceived*  
*If everyone is smiling, then everyone is truly happy*  
*Therefore, if everyone is smiling, then it is false that everyone is self-deceived*

We set up our deduction as follows:

- |    |                           |        |
|----|---------------------------|--------|
| 1) | $(x)(Hx \supset \sim Dx)$ | P      |
| 2) | $(x)(Sx \supset Hx)$      | P      |
| 3) | $Hu \supset \sim Du$      | 1 UI   |
| 4) | $Su \supset Hu$           | 2 UI   |
| 5) | $Su \supset \sim Du$      | 3,4 HS |
| 6) | $(x)(Sx \supset \sim Dx)$ | 5 UG   |

I note one more restriction on UG before we move on. When we are restoring a universal quantifier to a sentential function, all of the free variables must be replaced with the same bound variable, that is, with the same variable that is in the quantifier. For example, it is a valid inference to move from the sentential function “ $Su \supset Hu$ ” to the universal sentence  $(x)(Sx \supset Hx)$  or to  $(y)(Sy \supset Hy)$  but it is not valid to infer  $(x)(Sx \supset Hy)$ .

3. Our third quantificational rule of inference is called **Existential Generalization (EG)**. We can state this rule as follows:

<p><b>Existential Generalization (EG)</b></p> <p><math>Ga \therefore (\exists x)Gx</math></p> <p><math>Gu \therefore (\exists x)Gx</math></p> <p><i>“a” is an individual constant and “u” is a free variable</i></p>
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EG is free of any special restrictions, and simply tells us that we can validly infer from some particular singular sentence, for example, “Socrates is mortal” ( $M_s$ ), that “Something mortal exists,” which we symbolize as,  $(\exists x)Mx$ . Similarly, we can validly infer from the assumption that any randomly selected individual is mortal, that something is mortal. So EG allows us also to infer  $(\exists x)Mx$  from  $M_u$ . As in the case of UG, the rule of EG is a rule that adds quantifiers just as instantiation rules subtract them

4. The last quantificational rule is called **Existential Instantiation (EI)** □ The rule can be expressed as follows:

<p><b>Existential Instantiation (EI)</b></p> <p><math>(\exists x)Gx \therefore Ga</math></p> <p><i>“a” is an individual constant that has not occurred previously in the deduction; note that it is not a valid application of EI to infer <math>(\exists x)Gx \therefore Gu</math>, where “u” is a free variable</i></p>
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The reasoning behind the rule of EI is a little tricky. If something exists, then it follows that we can give that something a name. But this is only a hypothetical and arbitrary name, not a real name. In other words, from the fact that something exists it does follow that we can give that something a name. If it is true that something is mortal, we can give that something a name, for example, Peter. So from “ $(\exists x)Mx$ ,” EI allows us to infer “ $M_p$ .” An important restriction on this deduction is that the introduction of this individual constant “p” must be new, that is, it cannot have been used in any previous line of the deduction.

Clearly, we cannot validly deduce from the fact that something is mortal the inference that any randomly selected individual is mortal. All that validly follows is that at least one individual is mortal. In other words, the inference from “ $(\exists x)Mx$ ” to “ $\mu$ ” is not a valid application of the rule of EI.

OK then, let’s turn to a proof that involves all of our quantification rules:

- |     |   |         |
|-----|---|---------|
| 1)  | $(x)(Sx \supset Cx) \supset (\exists x)(Rx \bullet Hx)$ | P       |
| 2)  | $(x)(Sx \supset Hx) \bullet (x)(Hx \supset Cx)$         | P       |
| 3)  | $(x)(Sx \supset Hx)$                                    | 2 SIMP  |
| 4)  | $(x)(Hx \supset Cx)$                                    | 2 SIMP  |
| 5)  | $Sx \supset Hx$   | 3 UI    |
| 6)  | $Hx \supset Cx$   | 4 UI    |
| 7)  | $Sx \supset Cx$   | 5, 6 HS |
| 8)  | $(x)(Sx \supset Cx)$                                    | 7 UG    |
| 9)  | $(\exists x)(Rx \bullet Hx)$                            | 1, 8 MP |
| 10) | $Ra \bullet Ha$   | 9 EI    |
| 11) | $Ha$  | 10 SIMP |
| 12) | $(\exists x)(Hx)$                                       | 11 EG   |

When we turn to our *Exercise Workbook*, we will have plenty of practice in applying our new rules. Before we do so, we need to return to our Quantifier Negation rule to see how its various uses will figure in our predicate proofs. Our quantification rules of inference (UI, UG, EI, and EG), which allow us to take quantifiers away and to add them, do not apply if the quantified expressions we are working on are negated. Here is where the rule of **Quantifier Negation** comes into play. This rule allows us to take the negation sign away. Recall that every universally quantified sentence can be expressed as a negated existentially quantified sentence and vice versa. So, of course, if we negate a negated sentence, the negation is eliminated (the double negation rule of replacement called DN allows this). Consider the following two sentences: “ $\sim (x)(Sx \supset Cx)$ ” and “ $\sim (\exists x)(Rx \bullet Hx)$ ”

We cannot apply the rule of either UI or EI to these two assertions because they are negated. However, our rule of Quantifier Negation allows us to get rid of the negation sign by expressing the universal quantification as an existential quantification and vice versa. So “ $\sim (x)(Sx \supset Cx)$ ” is equivalent to the following existentially quantified sentence: “ $(\exists x) \sim (Sx \supset Cx)$ ”. After applying the rule of QN, we can now get rid of the quantifier by using EI to derive “ $\sim (Sa \supset Ca)$ ”.

Similarly, if a premise in our deduction is a negated existentially quantified assertion, we can express this as a universal sentence. That is, the sentence “ $\sim (\exists x)(Rx \bullet Hx)$ ” is logically equivalent to the sentence “ $(x) \sim (Rx \bullet Hx)$ ”. Again this application of the rule of QN allows us to remove the quantifier, and so clears the way for the application of the rule of UI.

Applying UI to“(x)~(Rx•Hx) we can validly infer ~(Ra•Ha) or alternatively ~(Ru•Hu).  
Before we turn to *Workbook Exercises* let’s see how the rule of QN works in a proof.

- |    |  |        |
|----|--|--------|
| 1) | $\sim (\exists x) \sim (Hx \supset \sim Dx)$ | P      |
| 2) | $(x)(Sx \supset Hx)$                         | P      |
| 3) | $(x)(Hx \supset \sim Dx)$                    | 1, QN  |
| 4) | $Hx \supset \sim Dx$                         | 3, UI  |
| 5) | $Sx \supset Hx$                              | 2, UI  |
| 6) | $Sx \supset \sim Dx$                         | 4,5 HS |
| 7) | $(x)(Sx \supset \sim Dx)$                    | 6, UG  |

# STUDY GUIDE

## Chapter 10 – Predicate Sentences and Predicate Proofs

### Quantification Rules of Inference

#### 1. Universal Instantiation (UI)

$$(\forall x)(Gx \therefore Ga)$$

$$(\forall x)Gx \therefore Gu$$

(“a” is an individual constant and “u” is a free variable)

#### 2. Universal Generalization (UG)

$$Gu \therefore (\forall x)Gx$$

(“u” is a free variable. Note this rule does not allow the inference:  $Ga \therefore (\forall x)Gx$ , where “a” is an individual constant)

#### 3. Existential Generalization (EG)

$$Ga \therefore (\exists x)Gx$$

$$Gu \therefore (\exists x)Gx$$

“a” is an individual constant and “u” is a free variable

#### 4. Existential Instantiation (EI)

$$(\exists x)Gx \therefore Ga$$

“a” is an individual constant that has not occurred previously in the deduction; note that it is not a valid application of EI to infer  $(\exists x)Gx \therefore Gu$ , where “u” is a free variable

Quantifier Negation (QN)
$\sim(\exists x)\sim Gx \equiv (\forall x)Gx$
$\sim(\forall x)\sim Gx \equiv (\exists x)Gx$
$\sim(\exists x)Gx \equiv (\forall x)\sim Gx$
$\sim(\forall x)Gx \equiv (\exists x)\sim Gx$

Translating A, E, I, O Categorical Sentences into Predicate Logic

A: All S are P:  $(\forall x)(Sx \supset Px)$

E: No S are P:  $(\forall x)(Sx \supset \sim Px)$

I: Some S are P:  $(\exists x)(Sx \bullet Px)$

O: Some S are not P:  $(\exists x)(Sx \bullet \sim Px)$