Chapter 8 - Sentential Truth Tables and Argument Forms

8.1 Introduction

The truth value of a given truth-functional compound sentence depends on the truth values of each of its components. One and the same sentence may be true if its components are all true and false if its components are all false. For example, the sentences, "The cat is on the mat and the dog is in the yard" ("C•D") is true if both the "C" and the "D" are true, but false if either "C" or "D" is false or if both are false.

A complete interpretation of this sentence will track every possible combination and permutation of truth values. Interpreting compound sentences that are not very complex is fairly easy. When these sentences become complex, interpreting them becomes more difficult. Of any given sentence, the logician should be able to say one of the following: that the sentence is (1) false under every possible interpretation; (2) true under every possible interpretation; or (3) true under some interpretations and false under others.

Logicians have given names to these three possible complete interpretations of sentences. A complete interpretation of a sentence will determine that it is a tautology, a contradiction, or a contingent sentence. We define these terms as follows:

Tautology:
A sentence that is true under every possible interpretation.
Contradiction:
A sentence that is false under every possible interpretation.
Contingent Sentence:
A sentence that is true under some interpretations and false under others.

The reason that it is important to know how to recognize these kinds of sentences is because knowing this can be of great help in evaluating arguments.

Accordingly, logicians have developed a technique for interpretation that will insure that every possible combination and permutation of truth values a given sentence can have is considered. This is the technique of interpreting sentences with truth tables. We have already introduced them informally in the last chapter. We used these simple tables to spell out the truth-conditions for conjunctive, disjunctive, conditional, and biconditional sentential forms. Now we need to explain this technique more formally so that we can use this method of tracking truth values to provide us with reliable and exhaustive interpretations of any given sentence or sentential form.

8.2 Constructing Truth Tables

The technique for constructing a truth table is rather simple and mechanical. There are three steps to follow. The first step in this construction process is to determine the number of columns of "T's" and "F's" that the table will have. To the far left of each truth table there will be columns of "T's" and "F's" with one column for each of the simple sentences in the compound sentence that is being interpreted. To the right of this we place one column for the sentence we are tracking. For example if we are interpreting the sentence, "(C ⊃ D) v ~(C v ~D)" we put one column for "C" and one column for "D." To the right of the column for "D" we put one column for the sentence we are tracking. So the number of columns is the number of simple sentences in the sentence that is being interpreted and one column for the sentence itself. So in our example our table will have three columns, one for C, one for D, and one for the sentence being tracked. The shell should look like this for our example.

C	D	(C⊃D) v ~(C v~D)

It is very important in this process of interpretation that you keep in mind which truth functional symbol is the main one, that is, the one that determines what kind of sentence it is that is being interpreted. This is important because each kind of sentence has different truth conditions. In our example, the main truth functional symbol is the wedge. That is, it is a disjunction. So we know it will be true if either side is true and false only if both disjuncts are false. A complete truth table will give us the complete interpretation of this disjunction, that is, every possible truth value it can have.

The second step is to determine the number of rows of "T's" and "F's" that our truth table will have. We call each row a row of interpretation. To do this, all we need to do is count the number of different simple sentences (that is, the number of different sentential letters) in the sentence that we are interpreting. Next, we plug that number into the following formula: 2^n (where n is the number of different sentence letters in the sentence and "2" represents the possible truth values, and of course there are only two, that is, true and false). For example, if our sentence has 3 different sentence e letters, our truth table will have 2^3 rows, that is, 8 rows of interpretation (2^4=16; 2^5=32; 2^6= 64, and so on). For our example we contains two different simple sentences (C and D) so we need to have 4 rows of interpretation. It should look like this:

C	D	(C ⊃ D) v ~(C v~D)

The third step is simply a matter of plugging in values to determine what truth value the sentence has in each row of interpretation. To accomplish this, we must fill in all the possible combinations and permutations of truth values that our simple sentences can have. In our example, the first two columns to the far left must represent every possible combination and permutation of T's and F's for "C" and "D". There is a mechanical procedure for insuring that we cover every possibility. Starting with "D" we simply alternate "T's" and "F's" as follows:

C	D	(C ⊃ D) v ~(C
	T	
	F	
	T	
	F	

Moving from right to left, we alternate T's and F's. That is, under "C," alternate T's and F's. When moving to the left we double the T's and F's as follow:

C	D	(C ⊃ D) v ~(C v ~D)
T	T	
T	F	
F	T	
F	F	

We just keep doubling as we move to the left in the table. So if we are tracking a sentence that has 3 different simple sentences, we would have to keep doubling as follows:

E	D	C

T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Of course, if we had 4 simple sentences we would have one more column of “T’s” and “F’s” and 16 rows of interpretation. The procedure for filling in the rows however stays the same: alternate, then double, then double again, and so forth.

Now we are ready to interpret our example sentence. To do this we simply make substitutions of truth values in the columns of the truth table to the right. After we have made these substitutions of truth values, we have a complete interpretation of the sentence and can tell if the sentence is a tautology, a contradiction or simply a contingent sentence. Again, this is determined by looking at the column of “T’s” and “F’s” under the main truth- functional symbol.

C	D	(C ⊃ D) v ~(C v ~D)
T	T	T
T	F	F
F	T	T
F	F	T

(In the process of making correct substitutions of truth values you must keep in mind what the truth-conditions are for each of our various compound sentences. That is, you are going to have to remember when conjunctions, disjunctions, conditionals, and so forth are true, and when they are false.)

Clearly, we see that the only case in which this disjunction is false is when C is true and D is false. This is shown in the third row of interpretation. On the other rows of interpretation, this disjunction is true. This means that the complete interpretation of this sentence shows that it not a tautology or a contradiction but a contingent sentence. That is, it is true on some interpretations and false on others. (Recall that a tautology has all “T’s” in the column under the sentence that is being interpreted, and a contradiction has all “F’s.” When we have a mix, we have a contingent sentence, as we do in this case.)

Here are some helpful hints to keep in mind in filling out the truth table.

- First, determine the main truth-functional symbol of the sentence you are interpreting. This will tell you what the truth conditions are for this sentence. For example, if you are interpreting a conditional sentence, you know it will only be false when the antecedent is true and the consequent false.
- On each row of interpretation, substitute truth values into one side or the other of the main truth-functional symbol of the sentence you are interpreting. You might want to work on the simplest side first. This might shorten this process. For example, if you are interpreting a conditional sentence and it clear that on this row of interpretation the consequent is true, then you know that on this row of interpretation this whole conditional must be true. In addition, if the sentence you are interpreting is a disjunction and in the row you are interpreting one side is true, then you know that the value of the whole disjunction on this row must be true.

8.3 Testing for Validity with Truth Tables

One of the most important concepts that we can learn in this course, and perhaps the most difficult, is that of validity. As we have said over and over in one way or another, an argument is valid if and only if it is impossible for the premises of that argument to be true and the conclusion false. We are now ready to see that this notion of validity has an interesting relation to conditional sentences.

The form of a conditional sentence, namely its “if/then” structure, exactly parallels the structure of an argument. Indeed, we read arguments as asserting that “if” the premises were true, “then” the conclusion must be true. So every argument has a kind of “if/then” or conditional structure.

There is something further to notice in this parallel. The only time that a conditional sentence is false is when the antecedent is true and the consequent is false. If we take the “if” part of an argument to parallel the antecedent of a conditional sentence and the “then” part to parallel the conclusion, we notice that the only case where a conditional sentence is false exactly parallels the only case in which an argument cannot be valid, that is, when its “if” part is true and its “then” part is false. Noticing these parallels allows us to come up with the following method for testing the validity of arguments with truth tables.

We will say that every argument may be expressed as a conditional sentence. (We must be careful here: a conditional sentence is not an argument, but it may express one.) We express an argument as a conditional sentence by making the antecedent of that sentence a conjunction of the premises of the argument that it is expressing. (If there is only one premise in the argument that we are expressing, then, of course, the antecedent of that conditional sentence will not be a conjunction.) Next, we make the conclusion of the argument that we are expressing the consequent of the conditional sentence.

Since no valid argument can have true premises and a false conclusion, and no true conditional sentence can have a true antecedent and a false conclusion, we can see that if an argument is valid, then the conditional sentence that expresses it must be a tautology. This gives us the following rule:

An argument is valid if and only if the conditional sentence that expresses it is tautological.

The first step in testing an argument with truth tables is to express that argument as a conditional sentence. Let’s see how this works with the following argument:

If I go to the movies then I will see Jane. I did go to the movies. Therefore, I saw Jane.

There are two premises in this argument. So to express this argument as a conditional sentence, we must conjoin these two premises and make them the antecedent of that sentence and make the conclusion the consequent of that sentence. Our expression of this argument as a conditional sentence then looks like this: $[(M \supset J) \bullet M] \supset J$. Now all we have to do is to construct a truth table to give a complete interpretation of this conditional sentence. That table would look like this:

J	M	$((M \supset J) \bullet M) \supset J$
T	T	
T	F	
F	T	
F	F	

Now we simply fill in the truth values under the main truth functional symbol. In this case I tis the horseshoe (⊃). That is, the sentence we are interpreting is a conditional.

J	M	$((M \supset J) \cdot M) \supset J$
T	T	T
T	F	T
F	T	T
F	F	T

As we know from our definitions above, this table shows that this conditional sentence is a tautology. Having determined that the conditional sentence that expresses the argument we are testing is tautological, we know that the argument is valid.

We now have a mechanical procedure for testing validity. There are three steps in the procedure. All we have to do is:

(1) Express the argument we are testing as a conditional sentence; (2) interpret it with a truth table; (3) Determine whether or not it is a tautology (it is, if and only if, there are all "T's" in the column below the sentence that is being tracked.)

If the conditional sentence is a tautology, the argument it expresses is valid; if it has even one "F" in the column under the conditional sentence being tested the argument it expresses is invalid.

8.4 A Short-Cut Test for Validity

The truth table method of testing for validity is fine so long as the number of different sentence letters is limited. In complicated arguments that involve many different sentient letters, the truth table method of testing for validity can become unwieldy. If, for example, we have an argument that involves 6 different sentence letters, our truth table will have 2^6=64 rows of "T's" and "F's." Of course the method will work in such complicated tables, but we might prefer a less cumbersome method if one is available. And fortunately one is available. We will call it the short-cut method.

If we correctly understand why an argument is valid if the conditional sentence that expresses it is tautological, then we can readily see how the short-cut method works. The only time that a conditional sentence is false is when the antecedent is true and the consequent is false. That combination of "T's" and "F's" is not possible if the conditional sentence is tautological. For if that combination did exist, then there would be an "F" in the interpretation of that sentence and it would not be tautological. With these things in mind, then our short-cut method is as follows:

Simply assign the consequent of the conditional sentence that expresses the argument we are testing the truth value of "F." (Now whatever the values are that you use to make the consequent false, these same values must be used when we make assignments to the antecedent.)

After we have assigned the consequent of the conditional sentence "F," we then see if there is any way to assign truth values to the sentence letters in the antecedent that will make it true. The values we assigned to the consequent to make it false, must be kept when assigning values to the antecedent.
If it is not possible to make the antecedent true when the consequent is false, the argument is valid. If it is possible to make the antecedent true when the consequent is false, then the argument is invalid.

OK, let's see how this short-cut method works. I have previously introduced two invalid argument forms, the fallacy of affirming the consequent and the fallacy of denying the antecedent. We can now demonstrate that these arguments are invalid. You can do this with a truth table or by the short-cut method. For example, consider the following argument that commits the fallacy of affirming

If I go to the movies then I will see Jane. I did see Jane.
Therefore, I went to the movies.

Express this argument as a conditional sentence as follows: "[(M ⊃ J) •J] ⊃ M. The conclusion of this argument (M) is here expressed as the consequent of this corresponding conditional sentence. Following our short-cut method, we simply assign this consequent, that is, "M" the truth- value "F." Now we see if there is any way that we can make assignments to the sentences in the antecedent that will make it true. Having assigned the value of F to M, we must keep that value as we make assignment in the antecedent. If we can do this, the argument is invalid; if not, it is valid. What if we make "J" true? If we do, then "M ⊃ J" will be true, and so the conjunction "(M ⊃ J) • J" will be true when the consequent "M" is false. This shows that the argument is invalid, for it shows that it is possible for the antecedent to be true when the consequent is false.

This short method is particularly useful when we have an argument with more than two or three simple sentences. Consider the following argument:

> If I go the movies then I will see Jane. If I go to the races, then I will see Sally. I will either go to the movies or to the races. Therefore, I will either see Jane or Sally.

To express this argument as a conditional sentence we must first symbolize the three premises and conjoin them to make the antecedent of the conditional sentence that will express the argument. That antecedent would be as follows: "[(M ⊃ J) • (R ⊃ S)] • (M v R)."

Now we make this expression the antecedent of the conditional sentence that expresses this argument and the conclusion its consequent, and we get this expression:

"{[(M ⊃ J) • (R⊃ S)] • (M v R)} ⊃ (J v S)"

Following the procedure for the short-cut method, we assign the consequent of this conditional sentence the truth value of "F," and then make truth value assignments to the other simple sentence letters in the antecedent, (keeping the assigned values we made in the consequent) to try to make the antecedent true. As it turns out, the only way to make the consequent false in this case is to make both "J" and "S" false, since this consequent is a disjunction and for a disjunction to be false, both disjuncts must be false. This means we are free to assign R or M whatever we like, keeping S and J as false. Having made this assignment, our only choices now are to assign truth values to "M" and "R." Remember, we are trying to make the antecedent true when the consequent is assigned the value "F." Since the antecedent is a conjunction, it can be true only if all of all of its conjuncts are true. Given that "J" is assigned the truth value of "F," the only way to make the first conjunct M ⊃ J true is to make "M" false. The same reasoning works with the second conditional "R ⊃ S." Given that "S" is assigned the value of false, the only way to make "R ⊃ S" true is to assign "R" the truth value of false. Having made these two assignments to "M" and "R" the final conjunct "M v R" of the antecedent, which is itself a disjunction, becomes false. So we see that there is no possible substitution of truth values for the simple sentences in this conditional sentence that would make it true when the consequent if false. Hence this conditional sentence is tautological and hence the argument that it expresses is valid.

8.5 Argument Forms

A particular argument contains sentences that have a particular content (as its premises and its conclusion). Such particular sentences are about this or that, e.g., cats on mats, dogs in yards, and sealing wax, and are symbolized with upper case letters that remind us of their content. We use these upper case letters that stand for simple sentences and refer to them as **sentence constants**. By contrast, we use lower case letters (p-z) to stand for sentence forms and refer to them as **sentence variables.** For example, "D" stands for a particular sentence, while "p" as a variable stands for any sentence whatsoever, even a compound one. That is the sentence variable "p" may stand for a disjunction, a conditional, and indeed, any sentence however complicated. In a similar vein, (p v q) stands for any disjunction whatsoever, (p ⊃ q) for any conditional sentence whatsoever, and so forth. For example, (p v q) stands for any disjunction whatsoever. This means that we can substitute the sentence D v G for the sentence variable p v q. As well, we can substitute more complex sentences for this sentence variable. For example, we can substitute the sentence (M ⊃ J) v R for the sentence variable p v q. In this case, p = (M ⊃ J) and q = R. We will say that the first sentence that we used to substitute for the sentence variable had the exact form as the variable and the second did not have that exact form. Both, however, are equally valid substitutions.

With this in mind, we can now note that any actual argument, which consists of a group of sentence constants, can always be expressed as a substitution instance of a more general argument form, that is, a form that consists of sentence variables. Consider the following particular argument that we will call "Argument A" and its sentential expression (Here we are introducing the following symbol "∴" to stand for "therefore.").

Argument A

If I go to the movies, then I will see Jane. I went to the movies. Therefore, I saw Jane.

We express Argument A as follows; M ⊃ J; M; ∴ J.

This actual argument is a substitution instance of, and indeed has the exact form of, the following argument that we have come to know as Modus Ponens. With the use of sentence variables we can express this valid argument more generally as follows: p ⊃ q; p; ∴ q

Accordingly, all of the valid arguments can be expressed is their most general form using sentence variables rather than sentence constants. We are now ready to express the valid arguments already introduced in their general argument forms.

Valid Forms

1. Modus Ponens (MP) p ⊃ q; p; ∴ q
2. Modus Tollens (MT) p ⊃ q; ~q; ∴ ~p
3. Hypothetical Syllogism (HS) p ⊃ q; q ⊃ r; ∴ p ⊃ r
4. Disjunctive Syllogism (DS) (p v q); ~q; ∴ p; or (p v q); ~p; ∴ q

Invalid Forms

1. Affirming the Consequent (AC) p ⊃ q; q; ∴ p
2. Denying the Antecedent (DA) p ⊃ q; ~p; ∴ q

If we determine that a particular argument is a substitution instance of one of these valid forms, then that argument is valid. The import of this can be generalized as follows:

Any particular argument is valid if is a substitution instance of a valid argument form.

This is very helpful since, as it happens, there are, surprisingly, only a few valid argument forms that we are likely to encounter amongst the thousands of different arguments that we commonly hear, read, and/or construct. In fact, we have already introduced some of these valid forms: Modus Pones, Modus Tollens, Disjunctive Syllogism, and Hypothetical syllogism. If you recognize a particular argument as being a substitution instance of one of these valid forms, then you know it is valid.

It is helpful to think of valid argument forms as rules of inference. To do this is to think of the valid argument forms as telling us what valid moves are open to us in formulating good deductive arguments. Take Modus Pones as an example. If I am assessing a particular argument that has one premise that is a conditional sentence, however, complex, and one premise that is the antecedent of that conditional sentence, then it is a valid move to deduce the consequent of that conditional sentence.

The same holds for Modus Tollens. If I am assessing a particular argument that has one premise that is a conditional sentence, however, complex, and one premise that negates the consequent of that conditional sentence, then it is a valid move to deduce the negation of the antecedent of that conditional sentence. We can think of Disjunctive Syllogism and Hypothetical Syllogism along the same lines.

Notice that what we are developing here a powerful technique for assessing validity. If any particular argument is a substitution instance of a valid argument form (we now know of 4 such forms, but we will introduce others as we proceed) then it is valid. If we run across an argument that is not a substitution instance of one a valid argument form, we do not know that it is invalid. Indeed, it may be valid or invalid. We can use our short-cut method to determine whether it is valid or not.

Nevertheless, this technique is not as helpful in determining invalidity. Indeed, finding that an argument is a substitution instance of an invalid form will not settle the question of validity unless it has the exact form of affirming the consequent or denying the antecedent. But if a particular argument is only a **substitution instance** of an invalid form it may be valid or invalid. In general: if an argument has the **exact form** of an invalid form, it will be invalid; if an argument is merely a substitution instance of an invalid form, and does not have that form as its specific form, then we do not know whether it is valid or not. That is, substitution instances of invalid forms may be valid or invalid. Consider the following example of an argument that is a substitution instance of the invalid form known as affirming the consequent, but it is nevertheless a valid argument:

If I go to the movies or go to the races, then I will go to the movies and go to the races. I go to the movies and I go to the

We can symbolize this argument as follows: {[(M v R) ⊃ (M•R)] • (M•R)} ⊃ (M v R) It should be clear to you that this argument is in fact a substitution instance of the invalid argument form of Affirming the Consequent, even though it does not have that exact form. However, in this case, the argument is valid. Indeed, we can easily show this with our short-cut method of determining validity.

Next, we assign the truth value of "F" to the consequent and try to make the antecedent true. If we can, the argument is invalid, if we cannot it is valid. In order to make the consequent here false, we must make both "M" and "R" false since the only way a disjunction can be false is for both disjuncts to be false. So we have to make the same assignments to the "M" and the "R" in the antecedent. Clearly the conjunction "M•R" is false. However, this conjunction is a conjunct of a larger conjunction. One false conjunct is sufficient to make a conjunction false. Hence, when the consequent of this conditional sentence is false, the antecedent cannot be true. Clearly then, the argument that this conditional sentence expresses is valid. As it happens, even though this argument is a substitution instance of the invalid form of affirming the consequent, it is nevertheless a valid argument.

Again, it is time to turn to the *Exercise Workbook*.

Study Guide

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Tautology:
A Sentence that is true under every possible interpretation.
Contradiction:
A Sentence that is false under every possible interpretation.
Contingent Proposition:
A Sentence that is true under some interpretations and false under others.

Valid Forms

1. Modus Ponens (MP) p ⊃ q; p; ∴ q
2. Modus Tollens (MT) p ⊃ q; ~q; ∴ ~p
3. Hypothetical Syllogism (HS) p ⊃ q; q ⊃ r; ∴ p ⊃ r
4. Disjunctive Syllogism (DS) (p v q); ~q; ∴ p; or (p v q); ~p; ∴ q

Invalid Forms

1. Affirming the Consequent (AC) p ⊃ q; q; ∴ p
2. Denying the Antecedent (DA) p ⊃ q; ~p; ∴ q

- **Valid Argument Forms:** It is sufficient to show that a particular argument is valid if it is a substitution instance of one of the four valid argument forms we have introduced so far (Modus Ponens, Modus Tollens, Hypothetical Syllogism, Disjunctive Syllogism).
- **Invalid Argument Forms:** It is sufficient to shoe that a particular argument is invalid if it has one of the two invalid argument forms as its exact form. (The two invalid argument forms are: (a) The Fallacy of Affirming the Consequent and (b) The Fallacy of Denying the Antecedent).